On some inequalities between prime numbers

Martin Maulhardt

July 2014

ABSTRACT. In 1948 Erdős and Turán proved that in the set of prime numbers the inequality $p_{n+2} - p_{n+1} < p_{n+1} - p_n$ is satisfied infinitely many times. By a new criteria of convergence of series we now prove that $p_{n+2k} - p_{n+k} \leq p_{n+k} - p_n$ is satisfied, for every k, infinitely many times. Additionally we introduce a new notion to measure how often this inequalities are satisfied and extend the scope of Erdős and Turán from prime numbers to any large set.

1. INTRODUCTION

The first proposition we prove is a new criteria of convergence of series of positive terms.

Theorem 1. Let $a_n : n \in N$ be a sequence of strictly increasing positive integers. Let the difference between the consecutive terms of this sequence be $d_n = a_{n+1} - a_n$ for $n = 1, 2, 3, \ldots$. If this sequence d_n is strictly increasing then the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is convergent.

Proof. Because the d_n are positive integers and because they are strictly increasing we have :

$d_1 \geq 1$
$d_2 \geq 2$
$d_3 \geq 3$
$\ldots$
$d_n \geq n$

By definition of d_n we find :

$d_1 = a_2 - a_1 \geq 1 \Rightarrow \ a_2 \geq 1 + a_1$
$d_2 = a_3 - a_2 \geq 2 \Rightarrow \ a_3 \geq 2 + a_2 \geq 2 + 1 + a_1$
$d_3 = a_4 - a_3 \geq 2 \Rightarrow \ a_4 \geq 3 + a_3 \geq 3 + 2 + 1 + a_1$

and in general we find that :

$a_n \geq 1 + 2 + 3 + \cdots + (n-1) + a_1$

Taking reciprocals and summing this inequalities over n we have :

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \leq \sum_{n=1}^{\infty} \frac{1}{a_1+1+2+....+(n-1)} \leq \sum_{n=1}^{\infty} \frac{1}{1+2+....+(n-1)} \leq \sum_{n=1}^{\infty} \frac{2}{n(n-1)} < \infty$$

Note. Theorem 1 remains true the differences d_n form an increasing sequence for $n \geq n_0$.

Example. Let F_n be the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$. Then the sequence d_n is $0, 1, 1, 2, 3, 5, 8, \ldots$. These differences d_n form an increasing sequence for $n \geq 2$. Then by a direct application of *Theorem 1* the series $\sum_{n=1}^{\infty} \frac{1}{F_n}$ is convergent.

2. A NEW PROOF OF $p_{n+2} - p_{n+1} \leq p_{n+1} - p_n$

In their paper *On some new questions on the distribution of prime numbers* Erdős and Turán proved that in the set of prime numbers the inequality $p_{n+2} - p_{n+1} < p_{n+1} - p_n$ is satisfied infinitely many times. By slightly weakening this inequality, namely, by writting $\leq$ instead of $<$ we will prove that this inequalities are satisfied infinitely many times by every large set. That is, this inequalities are not a consequence of the unique factorization of every natural number as a product of primes but a consequence of the divergence of the series of its reciprocals.

Definition. A set $A = \{a_n : n \in N\}$ is said to be a large set if the series of its reciprocals is divergent, i.e. $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$.

Examples.

1. The set N of natural numbers is a large set.

2. The set of prime numbers is a large set (Euler, 1737).

3. The set of square numbers is not a large set.

4. The set formed by the terms of the Fibonacci sequence is not a large set.

With this definition in mind we prove the following theorem.

Theorem 2. Let $a_n : n \in N$ be a sequence of strictly increasing positive integers. If the series $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$ then the inequality $a_{n+2} - a_{n+1} \leq a_{n+1} - a_n$ is satisfied infinitely many times.

Proof. Suppose that there is a last number n_0 that satisfies the inequality $a_{n_0+2} - a_{n_0+1} \leq a_{n_0+1} - a_{n_0}$. Then for every $n > n_o$ we have $a_{n+2} - a_{n+1} > a_{n+1} - a_n$. The first term of this inequality is d_{n+1} in *Theorem 1* and the second term is d_n. This implies that the sequence d_n is increasing for $n > n_0$. Thus, by *Theorem 1* the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is convergent. A contradiction.

Corollary 3. Let $p_1, p_2, p_3, \ldots p_n, \ldots$ be the sequence of prime numbers. Then we have $p_{n+2} - p_{n+1} \leq p_{n+1} - p_n$ infinitely many times.

Proof. The series $\sum_{n=1}^{\infty} \frac{1}{p_n}$ is divergent (Euler, 1737).

3. A GENERALIZATION OF $p_{n+2} - p_{n+1} \leq p_{n+1} - p_n$

Let us change the indexes in $p_{n+2} - p_{n+1} \leq p_{n+1} - p_n$. We obtain that infintely many times we have $p_{n+1} - p_n \leq p_n - p_{n-1}$. A question naturally arises : is it true that infinitely many times $p_{n+2} - p_n \leq p_n - p_{n-2}$? More generally, is it true that, for each fixed k, $p_{n+k} - p_n \leq p_n - p_{n-k}$ infinitely many times?

The main result of this section is to prove that both answers to these questions are in the affirmative. To prove them we need to define some notions.

Definition. Let $p_1, p_2, p_3, \ldots p_n, \ldots$ be the sequence of prime numbers. The sequence $p_1, p_3, p_5, \ldots p_{2n+1}, \ldots$ is the sequence of odd-indexed primes and the sequence $p_2, p_4, p_6, \ldots p_{2n}, \ldots$ is the sequence of even-indexed primes.

Proposition 4. Both the set $E = \{p_{2n} : n \in N\}$ and the set $O = \{p_{2n-1} : n \in N\}$ are large sets.

Proof. Suppose that the set O is not a large set. Then the series $\sum_{n=1}^{\infty} \frac{1}{p_{2n-1}} < \infty$. Using the comparison criteria of series of positive terms we also have $\sum_{n=1}^{\infty} \frac{1}{p_{2n}} < \infty$. But then $\sum_{n=1}^{\infty} \frac{1}{p_n} < \infty$. A contradiction. An analogous argument shows that E is large.

We are now ready to answer our first question.

Theorem 5. In the set of prime numbers the inequality $p_{k+2} - p_k \leq p_k - p_{k-2}$ is satisfied infinitely many times.

Proof. Take the set E of even-indexed prime numbers. By *Proposition 4* this set is large, i.e. the series $\sum_{n=1}^{\infty} \frac{1}{p_{2n}} = \infty$. Then by *Theorem 2* with $a_n = p_{2n}$ we find $p_{2n+4} - p_{2n+2} \leq p_{2n+2} - p_{2n}$ is satisfied infinitely many times and with a change of indexes $p_{2n+2} - p_{2n} \leq p_{2n} - p_{2n-2}$. This inequality can be written with $2n = k$ as $p_{k+2} - p_k \leq p_k - p_{k-2}$ which is what we wanted to prove.

The proof in the affirmative to the second question, namely that for any fixed k the inequality $p_{n+k} - p_n \leq p_n - p_{n-k}$ is satisfied infinitely many times is a simple extension of the arguments already used to prove it when $k = 2$. Instead of splitting the primes into even and odd indexed primes we split them into k large subsets $E_1, E_2, ...E_k$.

Let k be a fixed positive integer. For $1 \leq r \leq k$ define the set $E_r = \{p_{kn+r} : n \in N_0\}$. Each of these sets is a large set by an analogous argument used in the proof of *Proposition 4*. If we apply *Theorem 2* to any of these sets we obtain that infinitely many times $p_{n+k} - p_n \leq p_n - p_{n-k}$ which is the answer to question 2.

4. DENSITY OF THE INEQUALITY $p_{n+2} - p_{n+1} \leq p_{n+1} - p_n$

Let us index the triples of consecutive prime numbers in the following natural way :

$T_1 = (p_1, p_2, p_3)$
$T_2 = (p_2, p_3, p_4)$
$T_3 = (p_3, p_4, p_5)$
...
$T_n = (p_n, p_{n+1}, p_{n+2})$

Definition. A triple (a, b, c) of positive integers is said to be normal if $c - b \leq b - a$.

Examples. The triple $(7, 11, 13)$ is normal. The triple $(29, 31, 37)$ is not normal.

With this definition *Theorem 2* can be stated as follows : Every large set (in particular the set of prime numbers) contains infinitely many normal triples $T_n = (a_n, a_{n+1}, a_{n+2})$.

That these normal triples are infinite in number is a good result but it demands more precision. It could happen, for example, that these normal triples are $T_1, T_4, T_9, T_{16}, \ldots$. Or it could happen that they are $T_1, T_{100}, T_{10000}, T_{1000000} \ldots$. These situations are completely different from each other. We are trying to study how often do we have a normal triple.

Definition. Let $a_n : n \in N$ be a sequence of strictly increasing positive integers. We call the sequence a_n fundamental if it contains infinitely many normal triples of the form $T_n = (a_n, a_{n+1}, a_{n+2})$.

Example. The sequence p_n of prime numbers is fundamental.
The sequence $a_n = n^2$ is not fundamental.

We are now ready to explain informally our main result. That the sequence of prime numbers is fundamental means that there is no point at which the primes start to separate more and more. We will prove that if we consider the sequence whose terms are the indexes of the infinitely many normal triples of consecutive primes then this sequence is *ALSO* fundamental. This means that the normal triples T_n do not start to separate more and more as we go to infinity, the same as with the prime numbers themselves.

The following sequence will play a crucial role in our final result. Let $b_n : n \in N$ be the following fundamental sequence :

$b_n = 1, 3, 4, 6, 9, 10, 12, 15, 19, 20, 22, 25, 29, 34, 35, 37, 40, 44, \ldots$

whose first differences $d_n = b_{n+1} - b_n : n \in N$ are:

$d_n = 2, \quad 1, 2, 3, \quad 1, 2, 3, 4, \quad 1, 2, 3, 4, 5, \quad 1, 2, 3, 4, 5, 6, \quad \ldots$

where we have left a space to make our point clear.

Three *Lemmas* related to b_n are:

Lemma 6. The series $\sum_{n=1}^{\infty} \frac{1}{b_n}$ is convergent.

Proof. We have $\sum_{n=1}^{\infty} \frac{1}{b_n} =$

$1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{9} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \frac{1}{19} + \frac{1}{20} + \frac{1}{22} + \frac{1}{25} + \frac{1}{29} + \frac{1}{34} \cdots =$

$(1 + \frac{1}{3}) + (\frac{1}{4} + \frac{1}{6} + \frac{1}{9}) + (\frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \frac{1}{19}) + (\frac{1}{20} + \frac{1}{22} + \frac{1}{25} + \frac{1}{29} + \frac{1}{34}) \cdots <$

$(1 + 1) + (\frac{1}{4} + \frac{1}{4} + \frac{1}{4}) + (\frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10}) + (\frac{1}{20} + \frac{1}{20} + \frac{1}{20} + \frac{1}{20} + \frac{1}{20}) \cdots =$

$2 + \frac{3}{4} + \frac{4}{10} + \frac{5}{20} + \cdots = \sum_{n=1}^{\infty} \frac{n+1}{\frac{n^3+3n^2+2n}{6}} < \infty$

Lemma 7. The normal triples of the sequence b_n are : $T_1, T_4, T_8, T_{13}, T_{19}, T_{26}, \ldots$.

Proof. The proof is a simple observation of how the sequence b_n was constructed.

Lemma 8. The sequence of differences between two consecutive indexes of the normal triples is the sequence $r_n = 3, 4, 5, 6, 7, 8, \ldots$

Proof. Again this is by definition of the how the sequence b_n was constructed.

Theorem 9. Let p_n be the sequence of prime numbers. Let the triples of consecutive prime numbers be $T_n = (p_n, p_{n+1}, p_{n+2})$. Let $s_n : n \in N$ be the infinite sequence of the indexes of the normal triples T_n. Then the sequence $s_n : n \in N$ is fundamental.

Proof. Suppose that the sequence s_n is not fundamental. Then there is a number n_0 such that if $n \geq n_0$ the sequence of differences $r_n = s_{n+1} - s_n$ is strictly increasing.

We then have $r_{n_o+1} \geq 1, r_{n_0+2} \geq 2, r_{n_o+3} \geq 3, \ldots, r_{n_0+k} \geq k \ldots$. For simplicity of our argument let us change the indexes. We define $r_{n_0+3} = d_1, r_{n_0+4} = d_2, \ldots, r_{n_0+k+2} = d_k$. Then $d_1 \geq 3, d_2 \geq 4, d_3 \geq 5 \ldots, d_n \geq n+2 \ldots$.

We are now almost done. Let the triple that corresponds to the index s_{n_0+3} be $T_{s_{n_o+3}} = (q_o, q_1, q_2)$ and consider the sequence of primes starting at that q_0. We then have by how the sequence b_n was constructed:

$q_0 \geq 1$
$q_1 \geq 3$
$q_2 \geq 4$
$q_3 \geq 6$
$q_4 \geq 9$
$q_5 \geq 10$
$q_6 \geq 12$
$q_7 \geq 15$
$q_8 \geq 19$
$q_9 \geq 20$
$\ldots$
$q_n \geq b_{n+1}$
$\ldots$

Taking reciprocals and summing over n we obtain :

$\sum_{n=0}^{\infty} \frac{1}{q_n} \leq \sum_{n=1}^{\infty} \frac{1}{b_n} < \infty$

That is, the sum of the reciprocals of the primes starting at prime q_0 is convergent. And this would imply that the series $\sum_{n=1}^{\infty} \frac{1}{p_n} < \infty$.
A contradiction.
Thus the sequence s_n is fundamental, as claimed.

5. RELATION TO SOME OPEN PROBLEMS.

One of the most famous conjecture of Erdős and Turán is "Every large set contains arbitrary long arithmetic progresions". The Green-Tao theorem proves this for the set of prime numbers. This conjecture seems to be very difficult as no one (to our good faith) has yet even proved that "Every large set contains three elements in arithmetic progresion" let alone "arbitrary long arithmetic progresions".

Also Erdős and Turán proved in 1948 that infinitely many times we have in the set of prime numbers the inequality $p_{n+2} - p_{n+1} < p_{n+1} - p_n$.

We have proved in this paper that any large set satisfies infinitely many times the inequality $a_{n+2} - a_{n+1} \leq a_{n+1} - a_n$.

We can then conclude that any large set satisfies infinitely many times an analogous inequality to that found for prime numbers by Erdős and Turán, namely $a_{n+2} - a_{n+1} < a_{n+1} - a_n$ or every large set contains infinitely many triples of consecutive terms in arithmetic progresion, that is $a_{n+2} - a_{n+1} = a_{n+1} - a_n$ occur infinitely many times. But certainly at least one of these claims is true.

6. ACKNOWLEDGEMENT.

The author of this paper wishes to thank his two mentors Dr. Hector Guersenzvaig and Dr. Daniel Prelat and especially to his wife Julia for supporting him in infinite ways.

7. BIBLIOGRAPHY.

[1] P. Erdős, P. Turán: On some new questions on the distribution on prime numbers, Bull. Amer. Math. Soc. 54 (1948), 371–378 (MR9,498k; Zentralblatt 32,269.

[2] Green, Ben; Tao, Terence (2008), "The primes contain arbitrarily long arithmetic progressions", Annals of Mathematics 167 (2): 481–547, arXiv:math.NT/0404188, doi:10.4007/annals.2008.167.481.